ON SPECTRAL INVARIANCE OF NON-COMMUTATIVE TORI

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Abstract. Around 1980 Connes extended the notions of geometry to the non-commutative setting. Since then *non-commutative geometry* has turned into a very active area of mathematical research. As a first non-trivial example of a non-commutative manifold Connes discussed subalgebras of rotation algebras, the so-called *non-commutative tori*. In the last two decades researchers have unrevealed the relevance of non-commutative tori in a variety of mathematical and physical fields. In a recent paper we have pointed out that non-commutative tori appear very naturally in Gabor analysis. In the present paper we show that Janssen's result on good window classes in Gabor analysis has already been proved in a completely different context and in a very disguised form by Connes in 1980. Our treatment relies on non-commutative analogs of Wiener's lemma for certain subalgebras of rotation algebras by Gröchenig and Leinert.

1. Introduction

In [Gab46] D. Gabor proposed the following method for the transmission of a speech signal f. In modern language he discretized f into a sequence of bits, i.e. strings of the form "0100111". A natural way to transmit such strings is to send a pulse φ of length 1 at consequent time intervals and of size according to the amplitude of f, i.e. $f=\sum_{k=1}^m a_k\varphi(t-k)$. Now, a speech signal is a *band-limited* function, i.e. $\mathrm{supp}(\widehat{f})\subseteq[0,\theta]$ for some finite real θ, we observe

$$\widehat{f}(\omega)=\sum_{k=1}^m a_k e^{-2\pi ik\omega}\widehat{\varphi}(\omega)=\Big(\sum_{k=1}^m a_k e^{-2\pi ik\omega}\Big)\widehat{\varphi}(\omega).$$

Therefore, the essential support of f and of φ are equal which suggests a careful choice of the pulse φ. In practice we have to transmit more than one signal f, e.g. a conversation between a group of people. Gabor's brilliant idea was to shift each signal on a different frequency band. More precisely, if $f_1,...,f_n$ are the band-limited signals we want to transmit, then he suggested to send f_l on the l-th frequency band

$$f_l(t)=\sum_{k=1}^m a_{kl}e^{-2\pi il\theta t}\varphi(t-k).$$

Therefore, the transmission of all signals $f_1,...,f_n$ corresponds to

$$\sum_{l=1}^n f_l(t)=\sum_{l=1}^n\sum_{k=1}^m a_{kl}e^{-2\pi il\theta t}\varphi(t-k).$$

Date: January 21, 2006 .
1991 *Mathematics Subject Classification.* Primary 46L87, 47B48; Secondary 94A05.
Key words and phrases. Gabor analysis, non-commutative tori, spectral invariance.
The author was supported under the EU-project MEXT-CT-2004-517154.

Heisenberg's uncertainty principle implies that θ has to be at least greater than 1 since each signal occupies an area of size greater than or equal to 1.

The preceding observations indicates to choose for φ the Gaussian $\varphi(t) = e^{-\pi t^2}$, since the Gaussian is well-localized in the time-frequency plane. Therefore Gabor suggested to decompose an arbitrary signal $f \in L^2(\mathbb{R}^d)$ into a series of time-frequency shifts of a Gaussian over $\mathbb{Z}^{2d}$:

$$f = \sum_{k,l \in \mathbb{Z}^d} a_{kl} e^{2\pi i l} e^{-\pi(t-k)^2}. \tag{1}$$

The last equation is the first example of an *atomic decomposition* which in the last thirty years has led to a new field of mathematical research with an increasing literature, see the papers of Feichtinger and Gröchenig, [FG89a, FG89b], for the most general treatment of atomic decompositions. In honour of Gabor's lasting contribution we call decompositions of type (1) *Gabor series*. Gabor only gave heuristic arguments on the convergence of (1) for $f \in L^2(\mathbb{R}^d)$ but a rigorous analysis of (1) had to wait for Janssen's contribution [Jan81] of the year 1981. The main result of Janssen says that the convergence of (1) holds only in a weak sense of distributions, while Genossar and Porat [GP92] showed that the iterative approach may fail for certain L^2-functions. Out of Janssen's paper [Jan81] emerged the new mathematical field of *Gabor analysis*.

Gabor series (1) are built from so-called *time-frequency shifts*. Recall that the actions of these operators on $f \in L^2(\mathbb{R}^d)$ are given as follows:

(1) the *translation* operator by

$$T_x f(t) = f(t-x), \qquad x \in \mathbb{R}^d,$$

(2) the *modulation* operator by

$$M_\omega f(t) = e^{2\pi i t \cdot \omega} f(t), \qquad \omega \in \mathbb{R}^d,$$

(3) *time-frequency shifts* by

$$\pi(x,\omega) f(t) = M_\omega T_x f(t) = e^{2\pi i \omega t} f(t-x), \quad (x,\omega) \in \mathbb{R}^{2d}.$$

The time-frequency shifts $(x,\omega,\tau) \mapsto \tau M_\omega T_x$ for $(x,\omega) \in \mathbb{R}^{2d}$ and $\tau \in \mathbb{C}$ with $|\tau| = 1$ define the Schrödinger representation of the Heisenberg group, consequently the time-frequency shifts $\pi(x,\omega)$ for $(x,\omega) \in \mathbb{R}^{2d}$ are a projective representation of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. It is an important fact that time-frequency shifts satisfy the following composition law:

$$\pi(x,\omega)\pi(y,\eta) = e^{-2\pi i x \cdot \eta} \pi(x+y, \omega+\eta), \tag{2}$$

for $(x,\omega), (y,\eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Therefore, Gabor series (1) are often written in the following way:

$$f = \sum_{k,l \in \mathbb{Z}^d} a_{kl} M_l T_k e^{-\pi t^2}. \tag{3}$$

The computation of the coefficient $\mathbf{a} = (a_{kl})$ of (3) for a given signal $f \in L^2(\mathbb{R}^d)$ is one of the main problems in Gabor analysis. The solution of this problem is a non-trivial task since the building blocks in (3) are non-orthogonal, i.e. $\langle M_{l'}T_{k'}e^{-\pi t^2}, M_l T_k e^{-\pi t^2}\rangle \neq 0$ for $(k,l) \neq (k',l')$. Moreover, the coefficients $\mathbf{a}$ are in general not unique.

In section 2 we recall the solution of the above problem in terms of frames due to Daubechies, Grossmann and Meyer, [DGM86]. Furthermore we give the Janssen representation of a Gabor frame operator and point out the relevance of modulation spaces in Gabor analyis. In Section 3 we give a quick review of the central notions of non-commutative geometry. As application of the general principles we investigate rotation algebras and non-commutative tori. Finally we present the link between Gabor analysis and non-commutative tori. In Section 4 we define spectral invariant Banach and Fréchet algebras. We recall the fundamental results of Gröchenig and Leinert on window classes of Gabor frames. We close our discussion with a new approach to the spectral invariance of non-commutative tori in rotation algebras and indicate shortly Connes original argument of this important result.

2. Basics of Gabor analysis

Let $g \in L^2(\mathbb{R}^d)$ be a Gabor atom then $\mathcal{G}(g,\alpha,\beta) = \{\pi(\alpha k, \beta l)g : k,l \in \mathbb{Z}^d\}$ is called a *Gabor system*. Since the Balian-Low principle tells us that it is not possible to construct (this is in contrast to the situation with wavelets) an orthonormal basis for $L^2(\mathbb{R}^d)$ of this form, starting e.g. from a Schwartz function g, interest in Gabor frames arose. A milestone was the paper by Daubechies, Grossmann and Meyer, [DGM86], where the "painless use" of (tight) Gabor frames was suggested.

In our case the Gabor frame operator has the following form:

$$S_{g,\alpha,\beta}f = \sum_{k,l\in\mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l)g\rangle \pi(\alpha k, \beta l)g, \quad f \in L^2(\mathbb{R}^d).$$

A Gabor system $\mathcal{G}(g,\alpha,\beta)$ is called a *Gabor frame* if the Gabor frame operator $S_{g,\alpha,\beta}$ is invertible, i.e., if there exist some finite, positive real numbers A, B such that

$$A \cdot \mathrm{I} \leq S_{g,\alpha,\beta} \leq B \cdot \mathrm{I}$$

or equivalently,

$$A\|f\|^2 \leq \sum_{k,l\in\mathbb{Z}^d} \big|\langle f, \pi(\alpha k, \beta l)g\rangle\big|^2 \leq B\|f\|^2,$$

for all f in $L^2(\mathbb{R}^d)$. Gabor frames $\mathcal{G}(g,\alpha,\beta)$ allow the following reconstruction formulas for functions f in $L^2(\mathbb{R}^d)$

$$f = (S_{g,\alpha,\beta})^{-1}S_{g,\alpha,\beta}f = \sum_{k,l\in\mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l)g\rangle \pi(\alpha k, \beta l)(S_{g,\alpha,\beta})^{-1}g \tag{4}$$

$$= S_{g,\alpha,\beta}(S_{g,\alpha,\beta})^{-1}f = \sum_{k,l\in\mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l)(S_{g,\alpha,\beta})^{-1}g\rangle \pi(\alpha k, \beta l)g. \tag{5}$$

Due to its appearance in the reconstruction formulas $\gamma_0 := (S_{g,\alpha,\beta})^{-1}g$ is called the (canonical) *dual Gabor atom*. Another important observation is that the coefficients

in the reconstruction formula (4) are *not unique* and therefore there are other *dual atoms* $\gamma \in L^2(\mathbb{R}^d)$ with

$$S_{g,\gamma,\alpha,\beta} f := \sum_{k,l \in \mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l)\gamma \rangle \pi(\alpha k, \beta l) g.$$

Some authors call (g, γ) a *dual pair* of Gabor atoms if $S_{g,\gamma,\alpha,\beta} = \mathrm{I}$. In [WR90] the engineers Raz and Wexler characterized all dual pairs for finite Gabor system over cyclic groups. In the case of a Gabor frame $\mathcal{G}(g, \alpha, \beta)$ the following is equivalent (up to a technical condition):

(1) (g, γ) is a dual pair on $L^2(\mathbb{R}^d)$.
(2) $(\alpha\beta)^{-d} \langle M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} \gamma, M_{\frac{l'}{\alpha}} T_{\frac{k'}{\beta}} g \rangle = \delta_{kk'} \delta_{ll'}$.

Nowadays the preceding equivalent conditions are called the *Wexler-Raz biorthogonality relations*. This important result was obtained independently and with completely different methods by Ron-Shen, Janssen and Daubechies-H.J. Landau-Z. Landau in [RS97, Jan95, DLL95]. In the following we focus on Janssens approach since it provides an unexpected link to the work of Connes and Rieffel on non-commutative tori, see [Lu, Lu05].

Tolmieri and Orr proposed a new method for the calculation of the frame bounds of a Gabor system $\mathcal{G}(g, \alpha, \beta)$ in [TO95]. There method relies essentially on the following identity:

$$\sum_{k,l \in \mathbb{Z}^d} \langle f_1, \pi(\alpha k, \beta l) g_1 \rangle \langle \pi(\alpha k, \beta l) g_2, f_2 \rangle = \frac{1}{(\alpha\beta)^d} \sum_{k,l \in \mathbb{Z}^d} \langle g_2, \pi\big(\frac{k}{\beta}, \frac{l}{\alpha}\big) g_1 \rangle \langle \pi\big(\frac{k}{\beta}, \frac{l}{\alpha}\big) f_1, f_2 \rangle,$$

for f_1, f_2, g_1, g_2 in Schwartz space $\mathscr{S}(\mathbb{R}^d)$. Due to its great importance in his approach to the Wexler-Raz biorthogonality relations Janssen called the previous identity the *Fundamental identity of Gabor analysis* (FIGA). In [DLL95] it goes by the name *Weyl-Heisenberg identity* because many researchers call $\mathcal{G}(g, \alpha, \beta)$ a *Weyl-Heisenberg frame*. The work of Wexler-Raz related the original Gabor system $\mathcal{G}(g, \alpha, \beta)$ with another Gabor system $\mathcal{G}(g, \frac{1}{\beta}, \frac{1}{\alpha})$ over the lattice $\frac{1}{\beta}\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$. Neither Daubechies and her collaborators nor Janssen offered an explanation for the specific relation between the lattices $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and $\frac{1}{\beta}\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$ in their work. In [FK98] Feichtinger and Kozek were able to reveal a group theoretical interpretation for the pairing of the two lattices. They observed that the commutant of all time-frequency shifts $\{\pi(\alpha k, \beta l) : k, l \in \mathbb{Z}^d\}$ is given by all time-frequency shifts $\{\pi\big(\frac{k}{\beta}, \frac{l}{\alpha}\big) : k, l \in \mathbb{Z}^d\}$. Hence they were in the position to generalize the results in [DLL95, Jan95, RS97] to lattices Λ in elementary locally compact abelian groups because then the appropriate generalization of $\frac{1}{\beta}\mathbb{Z}^d \times \frac{1}{\alpha}\mathbb{Z}^d$ is the lattice Λ^0 defined as $\{\lambda^0 \in \mathbb{R}^d \times \widehat{\mathbb{R}^d} : \pi(\lambda)\pi(\lambda^0) = \pi(\lambda^0)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}$. Feichtinger and Kozek called Λ^0 the *adjoint lattice* of Λ. In their discussion of the adjoint lattice Feichtinger and Kozek pointed out the relevance of the symplectic Fourier transform in this context. Recently we recognized that Rieffel had already used the FIGA in his construction of projective modules over non-commutative tori, [Rief88]. In his proof of the FIGA he already applied the symplectic Fourier transform and he defined the adjoint lattice (in his terminology it is the dual lattice) in full generality 10 years before Feichtinger and Kozek. In [FL] we present a comprehensive treatment of

the FIGA and we considerable extend their range of applications to certain pairs of modulation spaces.

One of the main results in [Jan95] is the following interpretation of the FIGA for a Gabor system $\mathcal{G}(g,\alpha,\beta)$. By definition of $S_{g,\gamma,\alpha,\beta}$ the FIGA expresses that

$$\langle S_{g,\gamma,\alpha,\beta}f,h\rangle = \frac{1}{(\alpha\beta)^d}\langle S_{f,g,\frac{1}{\beta},\frac{1}{\alpha}}\gamma,h\rangle \tag{6}$$

holds for $f,g,h,\gamma\in\mathscr{S}(\mathbb{R}^d)$. More explicitly, the Gabor frame operator of $\mathcal{G}(g,\alpha,\beta)$ has the following representation

$$S_{g,\gamma,\alpha,\beta}f = \frac{1}{(\alpha\beta)^d}\sum_{k,l\in\mathbb{Z}^d}\langle\gamma,\pi(\frac{k}{\beta},\frac{l}{\alpha})g\rangle\pi(\frac{k}{\beta},\frac{l}{\alpha})f.$$

The preceding statement is the so-called *Janssen representation* of a Gabor frame operator. If we start with a Gabor atom $g\in L^2(\mathbb{R}^d)$ then the dual atom γ will have the same quality, i.e. $\gamma\in L^2(\mathbb{R}^d)$. Therefore, we have to impose some extra conditions to guarantee the validity of the Janssen representation. These conditions were already introduced by Tolimieri and Orr in [TO95]. If for a pair (g,γ) in $L^2(\mathbb{R}^d)$

$$\sum_{k,l\in\mathbb{Z}^d}|\langle\gamma,\pi(\frac{k}{\beta},\frac{l}{\alpha})g\rangle|<\infty,$$

then we say that the pair (g,γ) satisfies condition (A'). If $g=\gamma$, then g is said to satisfy condtion (A). The following theorem is a consequence of the preceding observations.

Theorem 2.1 (Janssen)**.** *Suppose that a pair of functions (g,γ) in $L^2(\mathbb{R}^d)$ satisfies condition (A') for a given lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$. Then*

$$S_{g,\gamma,\alpha,\beta} = \frac{1}{(\alpha\beta)^d}\sum_{k,l\in\mathbb{Z}^d}\langle\gamma,\pi(\frac{k}{\beta},\frac{l}{\alpha})g\rangle\pi(\frac{k}{\beta},\frac{l}{\alpha})$$

holds with absolute convergence in the operator norm.

In other words the great insight of Janssen was to express the Gabor frame operator for nice dual pairs (g,γ) as an absolutely convergent sum of time-frequency shifts from the adjoint lattice. We discuss Janssen's results on the structure of absolutely convergent sums of time-frequency shifts in section 3 because the notions of Connes and Rieffel provide the proper framework for these results.

The result of Tolimieri and Orr that each pair of Schwartz functions (g,γ) satisfies the condition (A') led to the search of other classes of function spaces with this nice property. In addition to the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ the modulation spaces $M^1_{v_s}(\mathbb{R}^d)$ provide us with a whole class of functions satisfying the property (A'). We now recall the definition and the basic properties of $M^1_{v_s}(\mathbb{R}^d)$.

In 1983 Feichtinger introduced a class of Banach spaces (see [Fei83,Fei03]), which allow a measurement of the time-frequency concentration of a function or distribution f on $\mathbb{R}^d$, the so called *modulation spaces*. For the measurement of the time-frequency concentration we choose the *Short-Time Fourier Transform* (STFT) because it is up to a phase-factor the representation coefficient of the Schrödinger

representation of the Heisenberg group. Concretely, if g is a window function in $L^2(\mathbb{R}^d)$ then the STFT of $f \in L^2(\mathbb{R}^d)$ is given by

$$V_g f(x,\omega) := \langle f, \pi(x,\omega)g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i\omega\cdot t}dt.$$

For functions f with good time-frequency concentration, e.g. Schwartz functions, the STFT can be interpreted as a measure for the amplitude of the frequency band near ω at time x. The properties of STFT depend crucially on the window function g.

Now we define a function or tempered distribution f to be an element of the *modulation space* $M^1_{v_s}(\mathbb{R}^d)$ if for a fixed g in Schwartz space $\mathscr{S}'(\mathbb{R}^d)$ the norm

$$\|f\|_{M^1_{v_s}} := \|V_g f\|_{L^1_{v_s}} = \int_{\mathbb{R}^{2d}} |V_g f(x,\omega)|(1+|x|^2+|\omega|^2)^{s/2}dxd\omega$$

is finite. Then $M^1_{v_s}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g. We always measure the $M^1_{v_s}$-norm with a fixed non-zero window $g \in \mathscr{S}(\mathbb{R}^{2d})$. If $s = 0$ then modulation space $M^1(\mathbb{R}^d)$ is the *Feichtinger algebra* which Feichtinger introduced in [Fei81].

In the following we state some of the properties of $M^1_{v_s}(\mathbb{R}^d)$, that are of interest in the later discussion.

(1) The dual space of $M^1_{v_s}(\mathbb{R}^d)$ is $M^\infty_{1/v_s}(\mathbb{R}^d)$ where the duality is given by

$$\langle f, h\rangle = \iint_{\mathbb{R}^{2d}} V_g f(x,\omega)\overline{V_g h(x,\omega)}dxd\omega,$$

for $f \in M^1_{v_s}(\mathbb{R}^d)$ and $h \in M^\infty_{1/v_s}(\mathbb{R}^d)$.

(2) $M^1_{v_s}(\mathbb{R}^d)$ is invariant under time-frequency shifts:

$$\|\pi(u,\eta)f\|_{M^1_{v_s}} \le Cv(u,\eta)\|f\|_{M^1_{v_s}} \quad \text{for} \quad (u,\eta) \in \mathbb{R}^{2d}.$$

(3) $M^1_{v_s}(\mathbb{R}^d)$ is invariant under Fourier transform.

The modulation spaces $M^1_{v_s}(\mathbb{R}^d)$ satisfy a weighted condition (A') which follows from the results in [FG89b].

Theorem 2.2. *If $g, \gamma \in M^1_{v_s}(\mathbb{R}^d)$, then*

$$\sum_{k,l\in\mathbb{Z}^d} \big|\langle \gamma, \pi(\frac{k}{\beta}, \frac{l}{\alpha})g\rangle\big|(1+\alpha^2|k|^2+\beta^2|l|^2)^{s/2} < C_{\alpha,\beta}\|g\|_{M^1_{v_s}}\|\gamma\|_{M^1_{v_s}}.$$

In particular, if $g, \gamma \in M^1(\mathbb{R}^d)$, then condition (A') is satisfied simultaneously for all $\alpha, \beta > 0$.

Finally we mention that the modulation spaces $M^1_{v_s}(\mathbb{R}^d)$ are the building blocks of the Schwartz class $\mathscr{S}(\mathbb{R}^d)$:

$$\mathscr{S}(\mathbb{R}^d) = \bigcap_{s\ge 0} M^1_{v_s}(\mathbb{R}^d). \tag{7}$$

By duality we get a useful description of the tempered distributions

$$\mathscr{S}'(\mathbb{R}^d) = \bigcup_{s \geq 0} M^{\infty}_{1/v_s}(\mathbb{R}^d). \tag{8}$$

All these statements on modulation spaces are well-known and the interested reader may find the proofs in Chapter 11 and Chapter 12 of [Gr01].

We close this section with another important theorem of Janssen. In his search for nice dual pairs (g, γ) Janssen took a Schwartz function g as Gabor atom for the Gabor system $\mathcal{G}(g, \alpha, \beta)$ and then he was able to show that the canonical dual window $\gamma_0 = S^{-1}_{g,\gamma,\alpha,\beta} g$ is also a Schwartz function.

Theorem 2.3 (Janssen). *Let $\mathcal{G}(g, \alpha, \beta)$ be a Gabor frame for $L^2(\mathbb{R}^d)$ with $g \in \mathscr{S}(\mathbb{R}^d)$. Then $\gamma_0 = S^{-1}_{g,\gamma,\alpha,\beta} g \in \mathscr{S}(\mathbb{R}^d)$.*

In other words if we start with a Gabor frame generated by a Schwartz function then the canonical dual window has the same quality. This result has substantially stimulated the research in Gabor analysis in the last decade. As an outcome of this research Gröchenig and his collaborators were led to the study of "nice" classes of involutive Banach algebras [FG05, GL, GL04].

3. Basics of non-commutative geometry

3.1. **Motivation.** Around 1980 Connes developed his new idea of non-commutative differential geometry which generalized the traditional differential and integral calculus, see [Con94]. Recall that unital and non-unital commutative C^*-algebras correspond to compact and locally compact spaces, respectively.

One of the central construction in harmonic analysis is the Pontrjagin dual of a locally compact abelian group. In the case of a discrete group Γ the Pontrjagin dual $\widehat{\Gamma}$ is a compact abelian group and the duality is given by the Fourier transform. This shows that the algebra of functions on $\widehat{\Gamma}$ can be identified with the reduced C^*-algebra of the group Γ,

$$C^*_r(\Gamma) \cong C(\widehat{\Gamma}).$$

If Γ is non-abelian Pontrjagin duality *no longer* works in the traditional sense, but the left hand side still makes perfect sense and "behaves" like the algebra of functions on the dual group. In other words, the Pontrjagin dual $\widehat{\Gamma}$ exists as a non-commutative space whose algebra of functions is the *twisted group C^*-algebra $C^*_r(\Gamma)$*. Therefore non-commutative geometry provides a natural generalization of Pontrjagin duality, in the sense that duals of discrete groups are non-commutative spaces.

In this sense unital group C^*-algebras provide us with a class of compact non-commutative spaces, e.g. the group C^*-algebra of a discrete non-abelian group. Consequently one thinks of a non-unital group C^*-algebra as a locally compact non-commutative space. The group C^*-algebra is the most prominent example of this class of objects. In non-commutative geometry spaces are thought as the same object if they are Morita equivalent which is a weakening of the notion of isomorphism for C^*-algebras, [Rief72, Rief74a, Rief74b, Rief76].

Furthermore Connes drew the attention of researchers in operator algebras to *smooth* subalgebras of C^*-algebras. The adjective "smooth" resembles the well-known fact that for a locally compact Hausdorff space M the space of smooth functions $C^\infty(M)$ is a dense Fréchet subalgebra of the C^*-algebra $C(M)$ of continuous functions on M. Like $C^\infty(M)$ smooth subalgebras of a C^*-algebra $\mathcal{A}$ are in general Fréchet subalgebras of $\mathcal{A}$. In the last two decades "smooth subalgebras" of C^*-algebras have been studied by many researchers, e.g. Badea, Bost, Cuntz, Ji and Schweitzer, see [Sch92, Sch93, Sch94].

In [GL, GL04] Gröchenig and Leinert have concentrated their efforts on non-commutative Banach algebras of a C^*-algebra $\mathcal{A}$ which may considered as the non-commutative analog of the k-times differentiable functions $C^r(M)$ on M. Later we expose their results for rotation algebras in detail. After this short digress on the underlying principles of non-commutative geometry we introduce the main operator algebras of our investigations: (1) rotation algebras and (2) non-commutative tori.

3.2. **Non-commutative tori.** Since the 1960's rotation algebras have been studied by Effros and his collaborators but the work [Con80, Rief81] of Connes and Rieffel on the structure of rotation algebras brought these algebras into the focus of researchers outside of operator algebras.

It is well-known that the C^*-algebra of continuous functions on the torus is the universal C^*-algebra generated by two commuting unitaries, which can be considered as the coordinate functions. This suggests considering the universal C^*-algebra $\mathbb{T}^2_\theta$ generated by unitaries U, V satisfying the commutation relation

$$UV = e^{2\pi i\theta} VU,$$

where the θ is a real number. $\mathbb{T}^2_\theta$ is called the *rotation algebra*. For an insightful exposition of rotation algebras we refer the reader to the monograph [Dav96]. We restrict ourselves to the connection between Gabor analysis and rotation algebras.

Let $\theta = \alpha\beta$ for positive reals α and β then the C^*-algebra generated by time-frequency shifts $\{\pi(\alpha k, \beta l) : k, l \in \mathbb{Z}^d\}$ is a representation of the rotation algebra $\mathbb{T}^2_\theta$ on $L^2(\mathbb{R}^d)$. Therefore, an element of $\mathbb{T}^2_\theta$ is given by

$$\sum_{k,l\in\mathbb{Z}^d} a_{kl}\pi(\alpha k, \beta l)$$

for a bounded complex-valued sequence $\mathbf{a} = (a_{kl})_{k,l\in\mathbb{Z}^d}$.

Lemma 3.1. *The representation of the rotation algebra by time-frequency shifts* $\{\pi(\alpha k, \beta l) : k, l \in \mathbb{Z}^d\}$ *is* **faithful** *on* $L^2(\mathbb{R}^d)$.

As a consequence it suffices to establish statements about the rotation algebra $\mathbb{T}^2_\theta$ for a dense subspace of $L^2(\mathbb{R}^d)$. In [Rief88] the interested reader finds a proof based on operator algebra methods and in [GL04] the reader finds another proof relying on time-frequency methods and Wiener amalgam spaces.

The faithfulness of the representation of $\mathbb{T}^2_\theta$ by time-frequency shifts $\pi(\alpha k, \beta l)$ from the lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ is essential for the results of Janssen, Gröchenig-Leinert and Rieffel, [GL04, Jan95, Rief88].

In [Con80] a *non-commutative torus* $\mathcal{A}^\infty$ was defined as the "smooth" elements of $\mathbb{T}^2_\theta$, i.e.

$$\mathcal{A}^\infty = \{A \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big) : A = \sum_{k,l} a_{kl} U^k V^l,\ \mathbf{a} = (a_{kl})_{k,l\in\mathbb{Z}^d} \in \mathscr{S}(\mathbb{Z}^{2d})\}.$$

$\mathscr{S}(\mathbb{Z}^{2d})$ is the space of sequences with rapid decay on $\mathbb{Z}^{2d}$. $\mathcal{A}^\infty$ was the first example of a *smooth* structure on a non-commutative C^*-algebra.

In 1995 Janssen described the non-commutative torus $\mathcal{A}^\infty$ in the representation of $\mathbb{T}^2_\theta$ by time-frequency shifts $\pi(\alpha k, \beta l)$. More precisely, in [Jan95] $\mathcal{A}^\infty$ is described as the following family of Banach algebras:

$$\mathcal{A}^1_s(\alpha,\beta) = \{A \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big) : A = \sum_{k,l\in\mathbb{Z}^d} a_{kl}\pi(\alpha k, \beta l),\ \mathbf{a} = (a_{kl})_{k,l\in\mathbb{Z}^d} \in \ell^1_s(\mathbb{Z}^{2d})\},$$

where $\ell^1_s(\mathbb{Z}^{2d})$ is the space of all sequences on $\mathbb{Z}^{2d}$ such that

$$\|\mathbf{a}\|_{1,s} := \sum_{k,l\in\mathbb{Z}^d} |a_{kl}|(1+\alpha^2|k|^2+\beta^2|l^2|)^{s/2} < \infty.$$

Consequently, the non-commutative tori $\mathcal{A}^\infty(\alpha,\beta)$ is just $\bigcap_{s\geq 0}\mathcal{A}^1_s(\alpha,\beta)$.

The Banach algebras $\ell^1_s(\mathbb{Z}^{2d})$ inherit from the commutation relations

$$\pi(\alpha k, \beta l)\pi(\alpha k', \beta l') = e^{2\pi i(k'\cdot l - k\cdot l')}\pi(\alpha k', \beta l')\pi(\alpha k, \beta l)$$

a *twisted product*

$$\big(\mathbf{a}\natural\mathbf{b}\big)(m,n) = \sum_{k,l\in\mathbb{Z}^d} a_{kl} b_{m-k,n-l} e^{2\pi i\theta(m-k)\cdot l}$$

for $\mathbf{a},\mathbf{b} \in \ell^1_s(\mathbb{Z}^{2d})$ and a *twisted involution*

$$a^*(k,l) = \overline{a_{kl}} e^{-2\pi i\theta k\cdot l}$$

for $\mathbf{a} = (a_{kl})_{k,l\in\mathbb{Z}^d}\ell^1_s(\mathbb{Z}^{2d})$.

Lemma 3.2. $\big(\mathcal{A}^1_s(\alpha,\beta),\natural,*\big)$ *is an involutive Banach algebra.*

We consider the involutive Banach algebra $\mathcal{A}^1_s(\mathbb{Z}^2)$ as the non-commutative analog of C^s-functions for the rotation algebra $\mathbb{T}^2_\theta$. Since the non-commutative analog of the differentiation operator is a *derivation* δ of a C^*-algebra $\mathcal{A}$, i.e.

$$\delta(AB) = \delta(A)B + A\delta(B),\ A,B \in \mathcal{A}.$$

On the non-commutative torus $\mathcal{A}^\infty$ we have a pair of commuting derivations

$$\delta_1\big(\sum a_{kl}U^kV^l\big) = 2\pi i\sum k a_{kl}U^kV^l$$

and

$$\delta_2\big(\sum a_{kl}U^kV^l\big) = 2\pi i\sum l a_{kl}U^kV^l$$

for $A = \sum a_{kl}U^kV^l$. Therefore the non-commutative analog of the Laplacian $\Delta = \delta_1^2+\delta_2^2$ acts on $\mathcal{A}^\infty$ by

$$\big(\delta_1^2+\delta_2^2\big)\big(\sum a_{kl}U^kV^l\big) = -4\pi^2\sum(k^2+l^2)a_{kl}U^kV^l.$$

By this reasoning we are tempted to consider those $A \in \mathbb{T}^2_\theta$ such that $\|\Delta^{s/2}A\|_{\mathrm{op}} < \infty$ as non-commutative analog of C^s for $s \in [0,\infty)$. Now, we suitably normalize our

"Laplacian" $\tilde{\Delta}$ and define the non-commutative analog of the potential operator by $I - \tilde{\Delta}$ which acts on $A \in \mathcal{A}^\infty$ in the following way,

$$\big(I - \widetilde{\Delta}\big)\big(\sum a_{kl} U^k V^l\big) = \sum (1 + k^2 + l^2) a_{kl} U^k V^l.$$

The preceding discussion suggests to think of the space of elements $A \in \mathbb{T}^2_\theta$ such that $\|(I - \widetilde{\Delta})^{s/2})A\|_{\mathrm{op}} < \infty$ for $s \in [0, \infty)$ as the non-commutative analog of the Sobolev spaces. In the case of the representation of $\mathbb{T}^2_\theta$ by $\{\pi(\alpha k, \beta l) : k, l \in \mathbb{Z}^d\}$ the space of $\|(I - \widetilde{\Delta})^{s/2})A\|_{\mathrm{op}} < \infty$ turns out to be $\mathcal{A}^1_s(\alpha, \beta)$.

Recall the fact that $f, g \in M^1_{v_s}(\mathbb{R}^d)$ implies that $\big\|\big(\langle f, \pi(\alpha k, \beta l) g\rangle\big)_{k,l}\big\|_{1,s} < \infty$, i.e.

$$A = \sum_{k,l \in \mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l) \in A^1_s(\alpha, \beta).$$

The preceding observation indicates the relevance of modulation spaces $M^1_{v_s}(\mathbb{R}^d)$ for Gabor analysis and non-commutative tori. We close this section with a characterization of $M^1_{v_s}(\mathbb{R}^d)$ by means of Gabor frames.

Theorem 3.3. *Assume that $g, \gamma \in M^1_{v_s}(\mathbb{R}^d)$ and that $S_{g,\gamma,\alpha,\beta} = I$ on $L^2(\mathbb{R}^d)$. Then*

$$f = \sum_{k,l \in \mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l)\gamma = \sum_{k,l \in \mathbb{Z}^d} \langle f, \pi(\alpha k, \beta l)\gamma\rangle \pi(\alpha k, \beta l) g$$

with unconditional convergence in $M^1_{v_s}(\mathbb{R}^d)$.

Furthermore, there are constants $A, A', B, B' > 0$ such that for all $f \in M^1_{v_s}(\mathbb{R}^d)$

$$A\|f\|_{M^1_{v_s}} \le \sum_{k,l \in \mathbb{Z}^d} |\langle f, \pi(\alpha k, \beta l) g\rangle| \le B\|f\|_{M^1_{v_s}}$$

and

$$A'\|f\|_{M^1_{v_s}} \le \sum_{k,l \in \mathbb{Z}^d} |\langle f, \pi(\alpha k, \beta l)\gamma\rangle| \le B'\|f\|_{M^1_{v_s}}.$$

The preceding result is just a special case of a general principle but it provides another reason for the search of nice classes of windows, see [Gr01].

4. Spectral invariance of Banach and Frechet algebras

In this section we proceed to state the non-commutative analogs of some traditional notions. In Connes' calculus a complex variable corresponds to an operator T on an infinite-dimensional Hilbert space $\mathcal{H}$ and a real variable is associated with a self-adjoint operator on $\mathcal{H}$. Consequently, the *spectrum* $\sigma(T)$ of T is the non-commutative analog of the range of a complex variable. The holomorphic functional calculus for operators in a Hilbert space gives meaning to $f(T)$ for any holomorphic function f defined on $\sigma(T)$ and only holomorphic functions act in that generality. This reflects the need for holomorphy in the theory of complex variables. Indeed, when the operator T is self-adjoint, $f(T)$ makes sense for any Borel function f on the line. In the following we present an extensive discussion of the holomorphic functional calculus. We begin with some well-known results of Wiener, Gelfand and Shilov on absolutely convergent Fourier series.

If we consider the Gabor system $\mathcal{G}(g,1,1)$ then the Banach algebras $\mathcal{A}^1_s(\alpha,\beta)$ and the Fréchet algebra $\mathcal{A}^\infty(\alpha,\beta)$ are commutative and involutive subalgebras of $\mathbb{T}^2$, i.e. the spaces of functions with absolutely convergent and rapidly decaying Fourier series. Now, an application of the Fourier transform allows the inversion of the Gabor frame operator, [FG97].

More concretely, Let $\mathcal{A}(\mathbb{T}^2)$ be the Banach algebra of all absolutely convergent Fourier series, i.e.

$$\mathcal{A}(\mathbb{T}^2) = \{f \in C(\mathbb{T}^2) : f(x,t) = \sum_{k,l\in\mathbb{Z}^d} a_{kl} e^{2\pi i(kx+lt)}, \sum_{k,l\in\mathbb{Z}^d} |a_{kl}| < \infty\}$$

with norm $\|f\|_{\mathcal{A}(\mathbb{T}^2)} = \sum_{k,l\in\mathbb{Z}^d} |a_{kl}|$. In the early 30's of the last century N. Wiener made an important observation about non-zero functions in $\mathcal{A}(\mathbb{T}^2)$. Namely, if $f \in \mathcal{A}(\mathbb{T}^2)$ is non-zero on $\mathbb{T}^2$, then $1/f$ is in $\mathcal{A}(\mathbb{T}^2)$, too. More precisely, there exists a sequence $b = (b_{kl})_{k,l\in\mathbb{Z}^d}$ such that $1/f = \sum_{k\in\mathbb{Z}^d} b_{kl} e^{2\pi i(kx+lt)}$. This fact goes by the name *Wiener's lemma*. This result of Wiener has many important applications and therefore $\mathcal{A}(\mathbb{T}^2)$ is often referred to as *Wiener's algebra*.

The first great success of Gelfand's theory of Banach algebras was a short elusive proof of Wiener's lemma. Later Naimark pointed out that Wiener's lemma is actually a statement about the pair of involutive Banach algebras $\mathcal{A}(\mathbb{T}^2) \subset C(\mathbb{T}^2)$ such that if a function $f \in \mathcal{A}(\mathbb{T}^2)$ is invertible in $C(\mathbb{T}^2)$ then $1/f$ is an element of $\mathcal{A}(\mathbb{T}^2)$. Naimark inspired by Wiener's result introduced the following notion.

Definition 4.1. *Let $\mathcal{A}$ be a subalgebra of the Banach algebra $\mathcal{B}$ with common unit I. Then $\mathcal{A}$ is called spectral invariant in $\mathcal{B}$ if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies $A^{-1} \in \mathcal{A}$. In this case $(\mathcal{A},\mathcal{B})$ is called a Wiener pair.*

Gröchenig and his collaborators call $\mathcal{A}$ *inverse-closed* in $\mathcal{B}$.

At the moment we draw some general conclusions from the definition of spectral invariance. We will focus on the spectral radius, the spectrum and holomorphic functional calculus for a general Wiener pair $(\mathcal{A},\mathcal{B})$ of Banach algebras. Since A is an element of $\mathcal{A}$ and $\mathcal{B}$ we can talk about the spectrum of A with respect to the algebra $\mathcal{A}$ and $\mathcal{B}$, respectively. Recall that the spectrum of A in $\mathcal{A}$ is defined as

$$\sigma_{\mathcal{A}}(A) = \{z \in \mathbb{C} : A - zI \text{ is not invertible in } \mathcal{A}\}. \tag{9}$$

An elementary argument yields the following lemma.

Lemma 4.2. *For $\mathcal{A} \subset \mathcal{B}$ with common unit I then*

$$\sigma_{\mathcal{B}}(A) \subset \sigma_{\mathcal{A}}(A).$$

In combination with the preceding observation we get the following innocent looking lemma which gives a justification to call a Wiener pair of Banach algebras $(\mathcal{A},\mathcal{B})$ spectral invariant.

Lemma 4.3. *Let $\mathcal{A} \subset \mathcal{B}$ be Banach algebras with common unit I. Then the following statements are equivalent:*

(1) *$(\mathcal{A},\mathcal{B})$ is a Wiener pair.*
(2) *$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$.*

In many situations we have to invert or to take a square root of an element A in a Banach algebra $\mathcal{A}$ with unit element I. In the early years F. Riesz, one of the pioneers of functional analysis, had the idea to build functions $f(T)$ of an compact operator $T \in \mathcal{B}(L^2(\mathbb{R}))$ in analogy to Cauchy's formula in complex analysis. Later this kind of reasoning had been continued by Wiener and substantially generalized by Dunford. Therefore the calculus goes by the names: *Riesz functional calculus*, *Dunford calculus* or *holomorphic calculus*.

More precisely, let $\mathcal{A}$ be a unital Banach algebra and $A \in \mathcal{A}$. Then

$$\mathrm{Hol}(A) = \{f : f \text{ is holomorphic on an open neighborhood } G \text{ of } \sigma_{\mathcal{A}}(A)\} \tag{10}$$

is the reservoir of functions which allows to form new elements $\tilde{f}(A)$ in the Banach algebra $\mathcal{A}$. Therefore, we choose a neighborhood G of $\sigma_{\mathcal{A}}(A)$ and a contour Γ of $\sigma_{\mathcal{A}}(A)$ in G. Then $f \in \mathrm{Hol}(A)$ with domain G allows to define

$$\tilde{f}(A) := \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz \tag{11}$$

as a Banach algebra valued integral. The basic results on Cauchy's formula in complex analysis imply that the definition of $\tilde{f}(A)$ is independent of Γ and that the integral exists as a Riemann integral and $\tilde{f}(A)$ is an element of $\mathcal{A}$. Recall, that $R(z, A) = (zI - A)^{-1}$ is called the *resolvent function* of A which is defined on the *resolvent set* $\rho_{\mathcal{A}}(A) = \mathbb{C} \backslash \sigma_{\mathcal{A}}(A)$. The resolvent function $R(., A)$ is analytic on $\rho_{\mathcal{A}}(A)$ and therefore $z \mapsto f(z)(zI - A)^{-1}$ is analytic from $G \cap \rho_{\mathcal{A}}(A)$ into $\mathcal{A}$.

The main theorem in this context says that for a fixed $A \in \mathcal{A}$. The mapping $A \mapsto \tilde{f}(A)$ is an algebra homomorphism and that this mapping is continuous from $\mathrm{Hol}(A)$ under uniform convergence on compact sets to $\mathcal{A}$ with the norm topology.

The following result is what one needs in the discussion of "nice" window classes of Gabor frames.

Theorem 4.4. *Let $\mathcal{A} \subset \mathcal{B}$ be Banach algebras with common unit I. If $(\mathcal{A}, \mathcal{B})$ is a Wiener pair, then the Riesz functional calculus for $\mathcal{A}$ coincides with the one for $\mathcal{B}$.*

Proof. Since $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$, the resolvent function $R(z, .)$ is defined in $\mathcal{A}$ and $\mathcal{B}$. Consequently, we get that $\mathrm{Hol}_{\mathcal{A}}(A)$ and $\mathrm{Hol}_{\mathcal{B}}(A)$ coincide. $\square$

The preceding discussion tells us that a Wiener pair $(\mathcal{A}, \mathcal{B})$ has many nice properties. But how can we decide if two Banach algebras $\mathcal{A} \subset \mathcal{B}$ with common unit I form a Wiener pair? That's actually the hard part in this topic and the only known tool is the so-called *lemma of Hulanicki* which relates the spectral invariance of $(\mathcal{A}, \mathcal{B})$ with the symmetry of $\mathcal{B}$.

The notion of symmetry of an involutive Banach algebra $\mathcal{A}$ with unit I is the generalization of the fact that a positive bounded operator T on a Hilbert space $\mathcal{H}$ has spectrum $\sigma(T) \subset [0, \infty)$.

Definition 4.5. *An involutive Banach algebra $\mathcal{A}$ with unit I is called* **symmetric**, *if $\sigma_{\mathcal{A}}(AA^*) \subset [0, \infty)$ for all $A \in \mathcal{A}$.*

An element $A \in \mathcal{A}$ is called **positive** if $A = CC^*$ for some $C \in \mathcal{A}$. In this terminology we can say that positive elements A in an involutive Banach algebra $\mathcal{A}$ with unit have "positive" spectrum. Therefore symmetry of an involutive Banach algebra measures how close $\mathcal{A}$ is to be a C^*-algebra.

Remark 4.6. *An involutive Banach algebra $\mathcal{A}$ with unit is symmetric if and only if $\sigma_{\mathcal{A}}(A) \subset \mathbb{R}$ for all $A = A^*$ in $\mathcal{A}$, if $A = A^*$ then A is* **hermitian**. *This fact is the* **Ford-Shiraly lemma** *and has been an open question for many years in the early days of normed algebras.*

The following theorem is the lemma of Hulanicki from the early 70's.

Theorem 4.7 (Lemma of Hulanicki)**.** *Assume that $\mathcal{B}$ is a symmetric Banach algebra and $\mathcal{A}$ a subalgebra of $\mathcal{B}$ with common unit I. Then $\mathcal{A}$ is spectral invariant in $\mathcal{B}$ if and only if the spectral radii for all $A = A^*$ with respect to $\mathcal{A}$ and $\mathcal{B}$ are equal, i.e.* $\mathrm{spr}_{\mathcal{A}}(A) = \mathrm{spr}_{\mathcal{B}}(A)$

A recent unpublished result by Leinert relates the notions of spectral invariance and symmetry of an involutive Banach algebra $\mathcal{A}$.

Theorem 4.8 (Leinert)**.** *Let $C^*(\mathcal{A})$ be the enveloping C^*-algebra of an involutive Banach algebra $\mathcal{A}$ with unit I then $\mathcal{A}$ is symmetric if and only if $\mathcal{A}$ is spectral invariant in $C^*(\mathcal{A})$.*

Now we have all notions and tools in hand to continue our investigations of "nice" window classes of a Gabor frame operator. Before we proceed we have to make some remarks. In [Jan95] the result that the canonical dual atom has the same quality as the Gabor atom was only proved under the additional assumption that $\theta = \alpha\beta$ is a rational number. At the end of the paper Janssen formulated the conjecture that the result is also true if θ is irrational. Therefore this conjecture was called the *irrational case conjecture*. The resolution of Janssen's conjecture was the great stimulus for the work of Gröchenig and Leinert on non-commutative analogs of Wiener's lemma for the irrational rotation algebra, [GL04].

Theorem 4.9 (Gröchenig-Leinert)**.** *$\mathcal{A}^1_s(\alpha,\beta)$ is spectral invariant in $\mathbb{T}^2_\theta$.*

Corollary 4.10. *$\mathcal{A}^1_s(\alpha,\beta)$ is stable under holomorphic calculus.*

If one recalls that the rotation algebra $\mathbb{T}^2_\theta$ is the enveloping C^*-algebra of $\mathbb{Z}^{2d}$ for the cocycle $\chi((k,l),(m,n)) = e^{2\pi i m\cdot l}$. In [GL04] the main result is the symmetry of the twisted convolution algebra $\ell^1_s(\mathbb{Z}^{2d}, \natural, ^*)$.

Consequently, the theorem of Gröchenig and Leinert is a special case of Theorem 4.8. At the beginning of the section we discussed Wiener's lemma and Wiener's algebra $\mathcal{A}(\mathbb{T}^2)$. Therefore the Banach algebra $\mathcal{A}^1_s(\alpha,\beta)$ is called the *non-commutative Wiener algebra*. As a further corollary of Theorem 4.9 we state the following result.

Theorem 4.11. *Let $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ a lattice in $\mathbb{R}^{2d}$ and let $g \in M^1_{v_s}(\mathbb{R}^d)$. If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then $S^\nu_{g,\alpha,\beta}g \in M^1_{v_s}(\mathbb{R}^d)$ for $\nu \in \mathbb{R}$. Especially, we have that for any $f \in L^2(\mathbb{R}^d)$*

$$f = \sum_{k,l\in\mathbb{Z}^d} \langle f, \pi(\alpha k,\beta l)g\rangle \pi(\alpha k,\beta l) S^{-1}_{g,\alpha,\beta}g = \sum_{k,l\in\mathbb{Z}^d} \langle f, \pi(\alpha k,\beta l)S^{-1/2}_{g,\alpha,\beta}g\rangle \pi(\alpha k,\beta l)S^{-1/2}_{g,\alpha,\beta}g.$$

Another consequence of Theorem 4.9 is the spectral invariance of $\mathcal{A}^\infty(\alpha,\beta)$. Since our weights $v_s(x,\omega)$ are submultiplicative for $s \in \mathbb{R}$, i.e. $v_s(x+y,\omega+\eta) \leq v_s(x,\omega)v_s(y,\eta)$. Therefore $\mathcal{A}^\infty(\alpha,\beta)$ is a Fréchet algebra with a scale of Banach algebras $\mathcal{A}^1_s(,\alpha,\beta)$ determined by a submultiplicative seminorm. Now we invoke an old result from the early days of operator algebras due to Michael, which says that the spectral invariance of each level $\mathcal{A}^1_s(\alpha,\beta)$ implies the spectral invariance of $\mathcal{A}^\infty(\alpha,\beta)$, [Mic52].

Corollary 4.12 (Connes-Janssen). *$\mathcal{A}^\infty(\alpha,\beta)$ is spectral invariant in the rotation algebra $\mathbb{T}^2_\theta$.*

We close the section with a short review of Connes' argument for the spectral invariance of $\mathcal{A}^\infty$ and $\mathbb{T}^2_\theta$.

We call a subalgebra $\mathcal{A}$ of a unital Banach algebra $\mathcal{B}$ **closed under holomorphic functional calculus** of A if it satisfies the following conditions:

(1) $\mathcal{A}$ is complete under some locally convex topology finer than the topology of $\mathcal{B}$;
(2) If $A \in \mathcal{A}$ and $f(A)$ is defined by the Riesz-Dunford integral, then $f(A) \in \mathcal{A}$.

In [Con80] Connes coined the notion of a *pre-C^*-algebra* for a subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ that is closed under holomorphic functional calculus.

The most well-known example of a pre-C^*-algebra is the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ whose C^*-completion is the C^*-algebra of continuous functions $C_0(\mathbb{R}^d)$ on $\mathbb{R}^d$. The characterization of the Schwartz class $\mathscr{S}(\mathbb{R}^d) = \cap_{s\geq 0} M^1_{v_s}(\mathbb{R}^d)$. In non-commutative geometry all pre-C^*-algebras are Fréchet algebras like $C^\infty(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d)$. Furthermore they arise as smooth vectors of a strongly continuous action.

Lemma 4.13. *Let $\mathcal{A}$ be a C^*-algebra, G a Lie group and ρ a strongly continuous action from G into the automorphisms $Aut(\mathcal{A})$ of $\mathcal{A}$. The dense subalgebra $\mathcal{A}^\infty$ of smooth elements under this action is a Fréchet pre-C^*-algebra.*

Proof. Now, for each $A \in \mathcal{A}$ the map $t \mapsto \rho_t(A)$ is continuous and $\mathcal{A}^\infty$ consists of those $A \in \mathcal{A}$ such that this map is smooth. Since for any $f \in \mathrm{Hol}(A)$ we have $f(\rho_t(A)) = \rho_t(f(A))$, the algebra $\mathcal{A}^\infty$ is closed under holomorphic functional calculus in $\mathcal{A}$. □

Now, Connes considered the following strongly continuous action. Let $\mathbb{T}^{2d}$ be the dual group of $\mathbb{Z}^{2d}$. Then $\mathbb{T}^{2d}$ has a natural dual action ρ on $\mathbb{T}^2_\theta$ given by

$$(\rho_t)(\mathbf{a})(k) = e^{2\pi i t\cdot k}a(k), \text{ for } \mathbf{a} \in \ell^1_s(\mathbb{Z}^{2d}), k \in \mathbb{Z}^{2d}, t \in \mathbb{T}^{2d}.$$

By the first lemma of Section 13 of [Con82], the space of smooth vectors for this action will be exactly $\mathscr{S}(\mathbb{Z}^{2d})$. Now Connes used that $\mathscr{S}(\mathbb{Z}^{2d})$ is closed under holomorphic functional calculus, see the appendix in [Con80]. Therefore $\mathcal{A}^\infty(\alpha,\beta)$ is spectral-invariant in $\mathbb{T}^2_\theta$.

5. Conclusion

The present investigation is aimed to link Janssen's outstanding contribution to Gabor analysis with non-commutative geometry, especially Connes's discussion of non-commutative tori. Our intention was to stress that a problem in applied mathematics gives rise to a deep result on non-commutative tori. More concretely, Gröchenig and Leinert's main insight is the equivalence of nice classes of Gabor atoms with the study of spectral invariant subalgebras of rotation algebras. But they were not aware of Connes's contribution to this problem and its connection to non-commutative geometry. Furthermore Gröchenig and Leinert restricted themselves to spectral invariant Banach algebras contrary to Connes's search for spectral invariant Fréchet subalgebras of rotation algebras. Finally we want to mention that Janssen's results have allowed to design new algorithms in Gabor analysis which lead to a significiant improvement of transfer rates for cellular phones or OFDM networks, see the contributions by Kozek and Strohmer in [FS98]. In other words *non-commutative tori have real-world applications.*

6. Acknowledgment

The author wants to thank K. Gröchenig for several discussions on spectral invariant Banach algebras and for his inspiring course "Banach algebra methods in applied mathematics" held at the University of Vienna during spring term 2005. Furthermore I express my gratitude for the kind invitation of D. Han and D. Larson to present some results of my Ph.D thesis at GPOTS 2005. Finally I want to thank H.G. Feichtinger for many suggestions and comments which improved the presentation considerably.

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